Approximation by Translates in Translation and Modulation Invariant Banach Spaces

Hans G. Feichtinger
Faculty of Mathematics
University of Vienna
hans.feichtinger@univie.ac.at

Anupam Gumber
Faculty of Mathematics
University of Vienna
anupam.gumber@univie.ac.at

***Abstract*—The main result of this note provides an alternative approach to the completeness result of a recent paper by V. Katsnelson ([5]). Instead of just using a collection of dilated Gaussians it is shown that the key steps of an earlier paper ([4]) by the authors, combined with the use of the non-vanishing of the Fourier transform allow us to show in a constructive manner that the linear span of the translates of a single function $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is a dense subspace of any Banach space satisfying certain double invariance properties. In fact, using well-established methods from the theory of Banach modules and time-frequency analysis, a much stronger statement is presented: for a given compact subset M in such a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one can construct a finite rank operator, whose range is contained in the linear span of finitely many translates of g, and which approximates the identity operator over M up to a given level of precision. There is a comprehensive list of examples of such spaces, and they do not have to be contained in $\boldsymbol{L}^2(\mathbb{R}^d)$.**

I. Introduction

For a given invariant Banach space of functions (or typically tempered distributions), this paper deals with the question that when can one assure that the closed linear span of the set of all translates of one suitable function coincides with the whole Banach space ? We show that well-established methods from the theory of Banach modules and time-frequency analysis allow to derive completeness results for the collection of translates of a given (test) function in a quite general setting.

The motivation for the current paper is the wish to demonstrate that the very specific results describing the density of the linear span of the set of all translates and dilates of the Gauss function as given in [5] can be generalized into several directions. In [5], V. Katsnelson investigated that for certain Hilbert spaces $\mathcal{H}$ of functions, continuously embedded into $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ satisfying a few additional (invariance) conditions has the property that the set of shifted and dilated Gaussian g span a dense subspace of $\mathcal{H}$, with $g(t) = \exp(-\pi t^2)$, $t \in \mathbb{R}$. This gives rise to the following questions: What are the relevant conditions for such a statement ? Is the Hilbert space structure important ? Can one have similar statements for Banach spaces of functions or distributions even if they are not inside of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$?

In the companion paper [4], we have shown that only the non-vanishing integral, i.e. without loss of generality the assumption $\int_{\mathbb{R}^d} g(x)dx = 1$ for the generating Schwartz function $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is sufficient in order to guarantee the density for a large variety of Banach spaces with double module structure. The present note can be seen as an alternative approach to the question of completeness of sets of translates of a given test function for a large variety of Banach space of tempered distributions. While the basic ideas show strong similarity to the arguments used in a recent paper by V. Katsnelson [5], we extend his results in several directions, both relaxing the assumptions and widening the range of applications. There is no need for the Banach spaces considered to be embedded into $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, nor is the Hilbert space structure relevant. Moreover, we also establish connections to modulation spaces and Shubin classes $\left(\boldsymbol{Q}_s(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{Q}_s}\right)$, showing that they are special cases of Katsnelson's setting (only) for $s \geq 0$.

As it turned out, the setting of the paper [2] appeared to be most appropriate, which is quite similar to the setting of so-called standard spaces as used in papers on compactness ([3]) or double module structures ([2]). We show that the linear span of the set of translates of a Schwartz function $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ satisfying $\widehat{g}(\xi) \neq 0, \forall \xi$, is dense in a large class of Banach spaces of tempered distributions called *minimal tempered standard spaces* (abbreviated as MINTSTA).

Definition I.1. *A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a* **MINTSTA**, *if it satisfies the following assumptions:*

1) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \subseteq \boldsymbol{B} \subseteq \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ *with continuous inclusions and* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ *is dense in* $\boldsymbol{B}$*;*
2) *the operators of translation and modulation T_x, $x \in \mathbb{R}^d$, and M_ξ, $\xi \in \mathbb{R}^d$, act continuously on $\boldsymbol{B}$ and there are $C_1, s_1, C_2, s_2 > 0$ such that $\|T_x f\|_{\boldsymbol{B}} \leq C_1(1+|x|)^{s_1}\|f\|_{\boldsymbol{B}}$ and $\|M_y f\|_{\boldsymbol{B}} \leq C_2(1+|x|)^{s_2}\|f\|_{\boldsymbol{B}}$, $\quad \forall x, \xi \in \mathbb{R}^d, f \in \boldsymbol{B}$.*

II. Main Results

For our main result, we first need to prove the following:

Theorem II.1. *If $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ is a translation invariant Banach space with*

$$\|T_x f\|_{\boldsymbol{B}} \leq w(x)\|f\|_{\boldsymbol{B}}, \quad \forall f \in \boldsymbol{B},$$

with submultiplicative Beurling weight function w. Then for any pair of functions $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ and $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ the product-convolution operator $f \mapsto k \cdot (g * f)$ defines a bounded operator from $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ into $\boldsymbol{L}^1_w(\mathbb{R}^d)$.

Our main result reads as follows:

Theorem II.2. *Given a relatively compact set M in any MINTSTA $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and some $g_0 \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ with $\widehat{g}_0(y) \neq 0$ for all $y \in \mathbb{R}^d$, we can show the following:*

For any given $\varepsilon > 0$ there exists some $\delta > 0$ such that for any δ-dense set $(x_i)_{i \in J}$ one can pick a finite subset $F \subset I$ and construct a finite rank operator T with range in the linear span of $S(g_0) := \{T_{x_i} g_0, i \in F\}$ with

$$\|T(f) - f\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M. \tag{1}$$

Remark II.3. *While the pure density of the linear span of the translates could be obtained in a non-constructive way via distributional and Fourier arguments the approach chosen allows to choose g more generally (it does not have to be smooth). A careful analysis shows that the assumption $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ in above result is convenient in the given setting, but in fact it is only required to assume $g \in \boldsymbol{B}_{1,w}(\mathbb{R}^d) = \boldsymbol{B} \cap \boldsymbol{L}^1_w(\mathbb{R}^d)$.*

For the two corollaries we continue to assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a MINTSTA.

Corollary II.4. *Assume that $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ satisfies $\widehat{g}(y) \neq 0$ for all $y \in \mathbb{R}^d$. Then for any finite set $M \subset \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ there is a sequence of finite rank operators T_n with range in the linear span of the finite set $S(g) := \{T_{x_i} g, i \in F\}$, such that $T_n(f) \to f$ in $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ for $n \to \infty$, for each $f \in M$. In particular, the linear span of all translates of g is dense in $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.*

For those who like to work with nets we can formulate our findings differently.

Corollary II.5. *Given $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ satisfying the Tauberian condition $\widehat{g}(y) \neq 0$ for all $y \in \mathbb{R}^d$ we can construct a net $(T_\alpha)_{\alpha \in I}$ of finite rank operators with range in the linear span of translates of g such that*

$$\lim_{\alpha \to \infty} T_\alpha f = f, \quad f \in \boldsymbol{B}.$$

References

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